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LETTER TO THE EDITOR

Bilinear Bäcklund transformation for the KdV equation with a source

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Abstract. A bilinear Bäcklund transformation for the KdV equation with a source is obtained by using the bilinear transformation method. Superposition formulae for solutions are then constructed which satisfy Bianchi's exchange property and it is shown that they have a simple structure when they are written in the bilinear forms. The inverse scattering transform for the equation is also derived from the Bäcklund transformation.

Recently, a new class of coupled soliton equations in two spatial and one temporal dimensions has been proposed [1]. They describe the interaction of a wavepacket of short waves with a single long wave on the x-y plane. It has also been shown that the proposed system of equations includes as special cases various known physical equations. The equation which we consider in this letter is one of these equations and it is written in the form

$$u_t + 6uu_x + u_{xxx} = -\int_{-\infty}^{\infty} \mathrm{d}k\, \nu(|\phi|^2)_x \tag{1}$$

$$\phi_{xx} + (u + k^2)\phi = 0 \tag{2}$$

where $u = u(x, t)$ and $\phi = \phi(x, t; k)$ are real and complex functions, respectively, k is a real parameter and $\nu = \nu(k, t)$ is a given real function. The subscripts appended to u and ϕ denote partial differentiations. The boundary conditions for u and ϕ are specified as

$$u \to 0 \qquad \text{as } |x| \to \infty \tag{3}$$

$$\phi = \phi_0\, \mathrm{e}^{\mathrm{i}kx} \qquad \text{as } x \to -\infty \tag{4}$$

where $\phi_0 = \phi_0(k, t)$ is a given function. Since when $\phi = 0$, (1) and (2) are reduced to the Korteweg–de Vries (KdV) equation, we call them the KdV equation with a source. This system of equations has already been shown to be integrable by means of the inverse scattering method [2, 3].

The purpose of the present paper is to derive a Bäcklund transformation (BT) of (1) and (2) and to clarify the structure of the BT. The mathematical tool employed here is the bilinear transformation method [4, 5]. In particular, it will be shown that superposition formulae for solutions have a simple structure when they are written in the bilinear forms.

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First of all, we bilinearize (1) and (2). Introduce the following dependent variable transformations

$$u = 2(\ln f)_{xx} \tag{5}$$

$$\phi = \phi \quad e^{ikx} g/f \tag{6}$$

where $f = f(x, t)$ and $g = g(x, t; k)$ are real and complex functions, respectively. Substituting (5) and (6) into (1) and (2) and using the boundary conditions (3) and (4), (1) and (2) are transformed into the following system of bilinear equations for f and g:

$$D_x(D_t + D_x^3) f\cdot f = -\int_{-\infty}^{\infty} \mathrm{d}k\, \nu|\phi_0|^2(|g|^2 - f^2) \tag{7}$$

$$(D_x^2 + 2\mathrm{i}kD_x) g\cdot f = 0. \tag{8}$$

Here, the bilinear operators D_t and D_x are defined by

$$D_t^m D_x^n g\cdot f = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n g(t, x) f(t', x')\bigg|_{\substack{t'=t\\ x'=x}}. \tag{9}$$

Let other solutions of (7) and (8) be f' and g'. We show that f' and g' are related to f and g by the following Bäcklund transformation:

$$(D_t + D_x^3 + \mu) f'\cdot f = \frac{\mathrm{i}}{4}\int_{-\infty}^{\infty} \mathrm{d}k\, \nu|\phi_0|^2 \left(\frac{g'^* g}{k - \mathrm{i}\lambda} - \frac{g' g^*}{k + \mathrm{i}\lambda}\right) \tag{10}$$

$$(D_x^2 - 2\lambda D_x) f'\cdot f = 0 \tag{11}$$

$$D_x g\cdot f' = -(\lambda + \mathrm{i}k)(g'f + gf') \tag{12}$$

$$D_x g'\cdot f = (\lambda - \mathrm{i}k)(g'f + gf'). \tag{13}$$

Here, λ and μ are real Bäcklund parameters which may be t-dependent in general and * denotes complex conjugate. First, we prove (12) and (13). For the purpuse, consider the quantity P defined by

$$P \equiv g'f'(D_x^2 + 2\mathrm{i}kD_x) g\cdot f - gf(D_x^2 + 2\mathrm{i}kD_x) g'\cdot f'. \tag{14}$$

Using the formulae for bilinear operators

$$abD_x c\cdot d - (D_x a\cdot b)cd = -D_x ad\cdot cb \tag{15}$$

$$abD_x^2 c\cdot d - (D_x^2 a\cdot b)cd = -D_x[(D_x a\cdot d)\cdot cb + ad\cdot (D_x c\cdot b)] \tag{16}$$

P is modified in the form

$$P = -D_x[(D_x g'\cdot f + \mathrm{i}kg'f)\cdot gf' + g'f\cdot (D_x g\cdot f' + \mathrm{i}kgf')]. \tag{17}$$

Substitution of (12) and (13) into (17) yields $P = 0$ due to the formula $D_x a\cdot a = 0$. This implies that if f and g are a pair of solutions of (8), f' and g' satisfy the same equation, verifying that (12) and (13) are BTs of (8).

Next, introduce the quantity Q defined below to prove (10) and (11).

$$Q \equiv f'^2\left[D_x(D t + D_x^3) f\cdot f + \int_{-\infty}^{\infty} \mathrm{d}k\, \nu|\phi_0|^2(|g|^2 - f^2)\right]$$
$$- f^2\left[D_x(D_t + D_x^3) f'\cdot f' + \int_{-\infty}^{\infty} \mathrm{d}k\, \nu|\gamma_0|^2(|g'|^2 - f'^2)\right]. \tag{18}$$

If we modify (18) with the aid of (15) and the formulae

$$b^2 D_t D_x a \cdot a - a^2 D_t D_x b \cdot b = 2D_x(D_t a \cdot b) \cdot ab \tag{19}$$

$$b^2 D_x^4 a \cdot a - a^2 D_x^4 b \cdot b = 2D_x(D_x^3 a \cdot b) \cdot ba + 6D_x(D_x^2 a \cdot b) \cdot (D_x b \cdot a) \tag{20}$$

and then substitute (11), (12) and (13) into the resultant expression, we obtain

$$Q = -2D_x\left[(D_t + D_x^3)f' \cdot f - \frac{\mathrm{i}}{4}\int_{-\infty}^{\infty} \mathrm{d}k\, \nu|\phi_0|^2\left(\frac{g'^* g}{k-\mathrm{i}\lambda} - \frac{g' g^*}{k+\mathrm{i}\gamma}\right)\right] \cdot f' f. \tag{21}$$

It now readily follows from (10) and the formula $D_x a \cdot a = 0$ that Q becomes zero, verifying that f' and g' satisfy (7) provided that f and g satisfy the same equation. Hence, we have completed the proof of the BT, (10)–(13). It should be remarked here that (13) follows from (8), (11) and (12) if we notice the identity

$$f' D_x^2 g \cdot f - g D_x^2 f' \cdot f = -2f_x D_x g \cdot f' + f(D_x g \cdot f')_x. \tag{22}$$

We now proceed to derive the superposition formulae in terms of bilinear variables. Let (f_1, g_1) and (f_2, g_2) be pairs of solutions of (7) and (8) generated by application of the BT, (11) and (12) to known solutions (f_0, g_0) with the Bäcklund parameters λ_1 and λ_2, respectively. Similarly, let (f_{12}, g_{12}) and (f_{21}, g_{21}) denote pairs of solutions of (7) and (8) obtained by application of the BT with the parameter γ_2 to (f_1, g_1) and with the parameter λ_1 to (f_2, g_2). The situation is represented schematically in figure 1 as a Bianchi diagram.

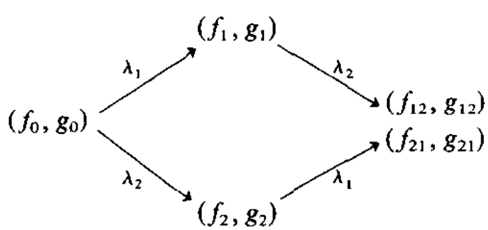

Figure 1. A Bianchi diagram.

From (11), (12) and the above definitions, the following system of equations must hold:

$$(D_x^2 - 2\lambda_1 D_x) f_1 \cdot f_0 = 0 \tag{23}$$

$$(D_x^2 - 2\lambda_2 D_x) f_2 \cdot f_0 = 0 \tag{24}$$

$$(D_x^2 - 2\lambda_1 D_x) f_{21} \cdot f_2 = 0 \tag{25}$$

$$(D_x^2 - 2\lambda_2 D_x) f_{12} \cdot f_1 = 0 \tag{26}$$

$$D_x g_0 \cdot f_1 = -(\lambda_1 + \mathrm{i}k)(g_1 f_0 + g_0 f_1) \tag{27}$$

$$D_x g_0 \cdot f_2 = -(\lambda_2 + \mathrm{i}k)(g_2 f_0 + g_0 f_2) \tag{28}$$

$$D_x g_2 \cdot f_{21} = -(\lambda_1 + \mathrm{i}k)(g_{21} f_2 + g_2 f_{21}) \tag{29}$$

$$D_x g_1 \cdot {}_{12} = -(\lambda_2 + \mathrm{i}k)(g_{12} f_1 + g_1 f_{12}). \tag{30}$$

We show that there exist solutions satisfying Bianchi's exchange property, namely $f_{21} = f_{12}$ and $g_{21} = g_{12}$. To do this, we first assume $f_{21} = f_{12}$. It then follows from $(23) \times f_2 f_{12} - (25) \times f_0 f_1$ and the formula

$$(D_x^2 a \cdot b)cd - ab D_x^2 c \cdot d = D_x[(D_x a \cdot d) \cdot cb + ad \cdot (D_x c \cdot b)] \tag{31}$$

that

$$D_x[(D_xf_1\cdot f_2)\cdot f_0f_{12}+f_1f_2\cdot(D_xf_{12}\cdot f_0)]-2\lambda_1D_xf_1f_2\cdot f_0f_{12}=0. \tag{32}$$

By the same way, we have from $(24)\times f_1f_{12}-(26)\times f_0f_2$

$$D_x[(D_xf_2\cdot f_1)\cdot f_0f_{12}+f_1f_2\cdot(D_xf_{12}\cdot f_0)]-2\lambda_2D_xf_1f_2\cdot f_0f_{12}=0. \tag{33}$$

Subtracting (33) from (32) gives

$$D_x[D_xf_1\cdot f_2-(\lambda_1-\lambda_2)f_1f_2]\cdot f_0f_{12}=0 \tag{34}$$

which means that

$$f_0f_{12}=c_1[D_xf_1\cdot f_2-(\lambda_1-\lambda_2)f_1f_2] \tag{35}$$

where c_1 is an arbitrary constant. Inversely, if we define f_{12} and f_{21} by (35), they satisfy (25) and (26) automatically. The relation (35) represents a superposition formula among four solutions f_0, f_1, f_2 and f_{12}. One also notes that another superposition formula is derived by adding (32) and (33) as

$$f_1f_2=c_2[D_xf_{12}\cdot f_0-(\lambda_1+\lambda_2)f_0f_{12}] \tag{36}$$

with c_2 being an arbitrary constant.

At this stage, we can prove $g_{21}=g_{12}$ under the conditions (35) and $f_{21}=f_{12}$. Indeed, it follows from $(29)\times f_0^2f_1-(30)\times f_0^2f_2$

$$R\equiv f_0^2f_1f_2(g_{12}-g_{21})=(g_2f_1-g_1f_2)f_0^2f_{12}+\left(\frac{f_1D_xg_2\cdot f_{12}}{\lambda_1+ik}-\frac{f_2D_xg_1\cdot f_{12}}{\lambda_2+ik}\right)f_0^2. \tag{37}$$

Substituting g_1 and g_2 obtained from (27) and (28) into (37) and using the formulae

$$(D_xa\cdot b)c=D_xa\cdot bc+abc_x \tag{38}$$

$$dD_x[a\cdot(D_xb\cdot c)]-cD_x[a\cdot(D_xb\cdot d)]=bD_x[a\cdot(D_xd\cdot c)] \tag{39}$$

we are led, after some calculations, to the expression

$$R=\frac{g_0}{(\lambda_1+ik)(\lambda_2+ik)}D_xf_0f_{12}\cdot[D_xf_1\cdot f_2-(\lambda_1-\lambda_2)f_1f_2]. \tag{40}$$

However, due to (35) and $D_xa\cdot a=0$, R vanishes identically. Since $f_0^2f_1f_2\neq 0$, we arrive at the required relation $g_{21}=g_{12}$. Thus, we have completed the construction of a commutative Bianchi diagram. If we use (11), (12) and (35), we can generate an infinite sequence of solutions $f_1, f_2, f_{12}, \ldots, g_1, g_2, g_{12}, \ldots$ starting from known solutions f_0 and g_0 by means of purely algebraic procedure. The bilinear BTs, (10)-(13), and bilinear superposition formulae (35) and (36) are easily transformed into ordinary forms by using (5), (6) and their primed counterparts. However, the explicit expressions are not written down here.

Finally, we conclude this letter by showing that the inverse scattering transform equations for (1) and (2) can be derived from the BT. For this, we introduce the wavefunctions $\psi=\psi(x,t)$ and $\chi=\chi(x,t;k)$ through the relations $\psi=f'/f$ and $\chi=\phi_0\,e^{ikx}g'/f$. It then turns out by eliminating f' and g' from (10)-(13) and using (5) and (6) that

$$\psi_t+\psi_{xxx}+3u\psi_x+\mu\psi=A\psi+B\psi_x \tag{41}$$

$$\psi_{xx}-2\lambda\psi_x+u\psi=0 \tag{42}$$

$$\chi=-(\lambda+ik)^{-1}[(\phi_x+\lambda\phi)\psi-\phi\psi_x] \tag{43}$$

$$\chi_x-\lambda\chi=(\lambda-ik)\phi\psi \tag{44}$$

with

$$A = -\frac{1}{4} \int_{-\infty}^{\infty} \mathrm{d}k \frac{\nu}{k^2+\lambda^2} \left(\frac{\partial}{\partial x} + 2\lambda \right) |\phi|^2 \tag{45}$$

$$B = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}k \frac{\nu |\phi|^2}{k^2+\lambda^2}. \tag{46}$$

One can confirm by direct calculation that the compatibility conditions of (41)–(44) yield (1) and (2).

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